SOLUTIONS OF EXERCISES.

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314*

A CYLINDER, diameter $2b$, intersects a sphere, diameter $2a$, the surface of the cylinder passing through the centre of the sphere. Required the part of the volume of the sphere contained by the cylinder.

[*Artemas Martin*].

SOLUTION.

Taking the origin at the centre of the sphere its rectangular equation is

$$x^2 + y^2 + z^2 = a^2, \tag{1}$$

and that of the cylindric hole is

$$x^2 + y^2 = 2bx. \tag{2}$$

Also,

$$V = \int\int\int dx\, dy\, dz. \tag{3}$$

Let $x = r \cos \varphi$, and $y = r \sin \varphi$; whence (1) and (2) become

$$r^2 + z^2 = a^2, \tag{4}$$

$$r = 2b \cos \varphi, \tag{5}$$

and

$$V = \int\int\int r\, dr\, d\varphi\, dz. \tag{6}$$

The limits of z are $-\sqrt{(a^2 - r^2)}$ and $+\sqrt{(a^2 - r^2)}$; of r, $2b \cos \varphi$ and 0; of φ, $\frac{1}{2}\pi$ and 0.

$$V = 2\int\int r\sqrt{(a^2 - r^2)}\, dr\, d\varphi, = -\frac{2}{3}\int (a^2 - r^2)^{\frac{3}{2}}\, d\varphi$$

$$= \frac{4}{3}\int_0^{\frac{1}{2}\pi} a^3 d\varphi - \frac{4}{3}\int_0^{\frac{1}{2}\pi} (a^2 - 4b^2 \cos^2\varphi)^{\frac{3}{2}}\, d\varphi. \tag{7}$$

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* The above solution is for the case $a > 2b$. The case $a < 2b$ remains to be treated. The case $a = 2b$ is old.—ED.

Let $\varphi = \frac{1}{2}\pi - \theta$; then $d\varphi = -d\theta$, $\cos\varphi = \sin\theta$, and (7) becomes

$$V = \frac{2}{3}\pi a^3 - \frac{4}{3}\int_0^{\frac{1}{2}\pi}(a^2 - 4b^2\sin^2\theta)^{\frac{3}{2}}\,d\theta$$

$$= \frac{2}{3}\pi a^3 - \frac{4}{3}a^2\int_0^{\frac{1}{2}\pi}\sqrt{(a^2 - 4b^2\sin^2\theta)}\,d\theta$$

$$+ \frac{16}{3}b^2\int_0^{\frac{1}{2}\pi}\sin^2\theta\sqrt{(a^2 - 4b^2\sin^2\theta)}\,d\theta$$

$$= \frac{2}{3}\pi a^3 + \frac{4}{3}a^3\int_0^{\frac{1}{2}\pi}\sqrt{(1 - e^2\sin^2\theta)}\,d\theta$$

$$+ \frac{16}{3}ab^2\int_0^{\frac{1}{2}\pi}\sin^2\theta\sqrt{(1 - e^2\sin^2\theta)}\,d\theta.$$

Putting $e = 2b/a$ we have

$$V = \frac{2}{3}a^3\pi - \frac{16}{9}a(a^2 - 2b^2)\,\mathrm{E}\left[\frac{1}{2}\pi, \frac{2b}{a}\right]$$

$$-\frac{4}{9}a(a^2 - 4b^2)\,\mathrm{F}\left[\frac{1}{2}\pi, \frac{2b}{a}\right].$$

When $b = \frac{1}{2}a$, $V = \frac{2}{3}a^3\left[\pi - \frac{4}{3}\right]$ [*Artemas Martin*].

321

A circle meets a hypocycloid of class 3 at six finite points. Show that the tangents to the hypocycloid at these six points touch a conic.

[*Frank Morley.*]

SOLUTION.

The equations of the hypocycloid in circular coordinates are

$$\left.\begin{aligned} x/c &= 2/t - t^2 \\ y/c &= 2t - 1/t^2 \end{aligned}\right\}, \tag{1}$$

t being a complex quantity of modulus 1. (See a paper on the Epicycloid, American Journal, Vol. XIII, No. 2).

Any circle is

$$xy + ax + \beta y + \gamma = 0.$$

Substituting from (1), we have a sextic to determine t, and we observe that the product of the roots is 1.

The tangent to (1) at t is

$$ux + vy + 1 = 0,$$

where

$$u = -t/c\,(1 + t^3), \quad v = -t^2/c\,(1 + t^3).$$

Let the line equation of a conic be

$$(A, B, C, F, G, H)(u, v, 1)^2 = 0.$$

Substituting for u, v in terms of t, we have again

$$\prod_1^6 t_r = 1,$$

Π denoting a product. This condition ensures that the six tangents touch a conic; and we saw that it holds if the points t are concyclic.

[*Frank Morley.*]

322

THE arc of a limaçon is shown in works on the Calculus to be equivalent to the arc of a certain ellipse. Show that the double point on the limaçon corresponds with Fagnani's point on the ellipse. [*W. B. Richards.*]

SOLUTION.

It is shown in works on the Calculus that

$$S = 2\int_0^{\frac{1}{2}\pi} \{(a + b)^2 \cos^2\varphi + (a - b)^2 \sin^2\varphi\}^{\frac{1}{2}}\, d\varphi,$$

is the quadrant of the ellipse on semi-axes $2(a + b)$ and $2(a - b)$, and also half of the whole length of the limaçon $r = a \cos\theta + b$.

It is well known that Fagnani's point divides the first arc into parts whose difference is $4b$, while the half difference between the two loops of the limaçon is also $4b$. [*W. H. Echols.*]

NOTE.—George R. Dean also points out that at Fagnani's point $\cos\varphi = \sqrt{\dfrac{a + b}{2a}} = \cos\tfrac{1}{2}\theta$ at the node of the limacon.—ED.

323

FOR solution see *Analyst*, Vol. I, No. 1, pp. 8–9. I proposed the problem in the *Schoolday Visitor Magazine* for May, 1872, nearly two years before Mr. Siverly used it in the *Analyst.* I had forgotten these facts when I sent it for publication in the ANNALS. [*Artemas Martin.*]